The Key Physical Parameters Governing Frictional Dissipation in a Precipitating Atmosphere

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ABSTRACT

Precipitation generates small-scale turbulent air flows—the energy of which ultimately dissipates to heat. The power of this process has previously been estimated to be around 2–4 W m\(^{-2}\) in the tropics: a value comparable in magnitude to the dynamic power of global atmospheric circulation. Here it is suggested that the true value is approximately half the value of this previous estimate. The result reflects a revised evaluation of the mean precipitation pathlength \(H_p\). The dependence of \(H_p\) on surface temperature, relative humidity, temperature lapse rate, and degree of condensation in the ascending air were investigated. These analyses indicate that the degree of condensation, defined as the relative change of the saturated water vapor mixing ratio in the region of condensation, is a major factor determining \(H_p\). From this theory the authors develop an estimate indicating that the mean large-scale rate of frictional dissipation associated with total precipitation in the tropics lies between 1 and 2 W m\(^{-2}\) and show empirical evidence in support of this estimate. Under terrestrial conditions frictional dissipation is found to constitute a minor fraction of the dynamic power of condensation-induced atmospheric circulation, which is estimated to be at least 2.5 times larger. However, because \(H_p\) increases with increasing surface temperature \(T_s\), the rate of frictional dissipation would exceed the power of condensation-induced dynamics, and thus block major circulation, at \(T_s \geq 320\) K in a moist adiabatic atmosphere.

1. Introduction

Understanding the physics of a moist atmosphere and capturing it in theoretical concepts is a major challenge for climate science (Schneider 2006; Schiermeier 2010; Cotton et al. 2011). Among the complications introduced by water vapor are the various influences of precipitation on atmospheric motion. One specific aspect is that precipitation generates small-scale air turbulence around the falling condensate particles. The energy for this turbulent air motion derives from the potential energy of the rain drops or ice particles (i.e., “hydrometeors”) in the gravitational field of Earth and is ultimately dissipated to heat. If this energy were not converted to turbulent kinetic energy of the air, the hydrometeors would continue to accelerate as they fall. But air exerts a drag force that prevents this acceleration. This force grows with increasing size of the hydrometeor and its velocity \(W\) relative to the surrounding air. Thus, as the condensate particle is accelerated by gravity, this opposing force grows until it equals the weight of the particle. Acceleration ceases at this terminal velocity \(W_t\). Since hydrometeors fall near to their terminal velocities for most of the duration of their falls, the mean drag force acting on them over this period is approximately equal to their weight.

Consider a column of moist air as a mixture of dry air and water vapor, which we denote here using subscripts...
and \( v \), respectively, standing for dry air and water vapor. In hydrostatic equilibrium,

\[
\frac{\partial p}{\partial z} = \rho g,
\]

where \( p = p_d + p_v \) is air pressure, \( \rho = \rho_d + \rho_v \) is air density, and \( g \) is the acceleration due to gravity. There is no available potential energy in such a column. We now cool this column in such a manner that some water vapor condenses. Two types of potential energy become available: the potential energy of droplets in the gravitational field and the potential energy of any nonequilibrium air pressure gradient that may have formed upon condensation. We recently proposed that the release of the second type of potential energy (i.e., that associated with the nonequilibrium gradient of saturated water vapor) is a major driver of atmospheric circulation on Earth (Makarieva and Gorshkov 2007, 2010; Gorshkov et al. 2012; Makarieva et al. 2013b). Since the change of gas pressure and the frictional dissipation associated with droplets are inseparable aspects of condensation, it is important to estimate and contrast the power of both processes.

Another reason for studying frictional dissipation associated with precipitation is that it is a major dissipative process in the atmosphere of Earth (Pauluis et al. 2000). We require a thorough theoretical analysis of all factors that can possibly determine its value especially in a changing climate. In this paper we first examine how the power \( D \) of precipitation-related frictional dissipation can be estimated from basic atmospheric parameters. We then compare our results to those of Pauluis et al. (2000) and argue that our estimates are more consistent with both theory and data, including the recent estimate of \( D \) by Pauluis and Dias (2012) from satellite-derived tropical rain rates. We show that \( D \) grows with increasing surface temperature and estimate the critical temperature when the power of frictional dissipation equals the power of condensation-induced dynamics, such that the latter may cease.

## 2. Basic formulas

The power of dissipation of energy per unit area (\( W \text{ m}^{-2} \)) associated with small-scale turbulence around hydrometeors can be written as

\[
D = \int_0^\infty W F_c dz = \int_0^\infty W_i \rho \gamma dz.
\]

Here \( W = w_c - w > 0 \) is the mean velocity of hydrometeors relative to the air; \( w_c \) and \( w \) are the vertical velocities of condensate particles and air relative to Earth’s surface, respectively; \( F_c \) is the turbulent drag force per unit air volume exerted by air on hydrometeors; \( \rho_c = N_c m \) is condensate density; \( N_c \) is the number of hydrometeors per unit volume; and \( m \) is their mean mass. The second equality in Eq. (2) takes into account our assumption that hydrometeors are falling at \( w_c \), such that the drag force acting on a droplet is equal to its weight. For a constant \( W_i \), Eq. (2) simply represents the product of \( W_i g \) and the total amount of condensate in the atmospheric column.

Equation (2) is not suited for a theoretical analysis. The distribution of \( W_i \), which depends strongly on particle size, is poorly known. The amount of condensate in the atmosphere varies greatly in time and space, from a few kilograms per square meter in severe storms to less than 100 g m\(^{-2}\) under normal conditions (e.g., Jiang et al. 2008; O’Dell et al. 2008; Wood et al. 2002). Nonetheless, as we shall now show, it is possible to modify Eq. (2) to exclude these uncertain parameters.

We first consider the case when the terminal velocity of hydrometeors is much larger than air velocity, \( W_t \gg |w| \). Neglecting for now reevaporation of condensate in the column, we assume that all condensate that has formed in the course of the ascent of moist air precipitates to the ground. In such a case the power becomes

\[
D = \rho c_s W_t g H_p = \rho g H_p.
\]

Here, \( \rho_c \) and \( \rho_a \) are respectively the condensate density and the terminal velocity of hydrometeors near the ground surface, precipitation pathlength \( H_p \) is the mean height from which the hydrometeors fall, and \( P_s = \rho_c \rho_a W_t \) is precipitation measured at the surface (kg H\(_2\)O m\(^{-2}\) s\(^{-1}\)). When a hydrometeor falls to the ground from height \( H_p \), the potential energy lost per unit mass is \( gH_p \). Precipitation \( P_s \) tells us how much water hits the ground per unit surface area per unit time, so \( D = \rho_g H_p \) gives the total rate of potential energy loss by all hydrometeors in the column.

At \( W_t \gg |w| \) there is no upward transport of condensate that originated in the lower atmospheric layers: the hydrometeors fall to the ground from where they were formed. Rate of condensation \( S \) (mol H\(_2\)O m\(^{-2}\) s\(^{-1}\)) in the ascending air is

\[
S = -w \left( \frac{\partial N_v}{\partial z} - \gamma \frac{\partial N}{\partial z} \right) = -w N \frac{\partial \gamma}{\partial z} > 0; \quad \gamma = \frac{N_v}{N} = \frac{\rho_v}{\rho},
\]

where \( N_v \) and \( N \) (mol m\(^{-3}\)) are the molar densities of water vapor and air (Makarieva and Gorshkov 2010; Gorshkov et al. 2012). The value of \( S \) differs from zero only in a certain layer \( z_1 \leq z \leq z_2 \), where the relative
humidity is close to unity and the water vapor is saturated. Precipitation pathlength is then equal to the mean height of condensation:

\[
H_p = \int_0^{\infty} \frac{S(z) dz}{\gamma^s(z_1) z_1 - \gamma^s(z_2) z_2 + \int_{z_1}^{z_2} \gamma^s(z) dz} \gamma^s(z_1) - \gamma^s(z_2).
\]

Here, \(z_1\) and \(z_2\) are the heights of the lower and upper boundaries of the condensation layer (\(S = 0\) for \(z < z_1\) and \(z > z_2\)) and \(\gamma = \gamma^s = p^v/p\) is equal to the ratio of saturated water vapor partial pressure \(p^v\) to air pressure \(p\). (Note that \(\gamma^s = (M_d/M_v) r^s / [1 + r^s (M_d/M_v)] \approx (M_d/M_v) r^s\), where \(r^s = \rho^s / \rho_d\) is the saturated water vapor mixing ratio and \(M_v\) and \(M_d\) are the molar masses of the water vapor and dry air, respectively. Thus, replacing \(\gamma^s\) by \(r^s\) in Eq. (5) will not significantly affect the estimate of \(H_p\) for \(r^s \ll 1\).) Condensation rate \(S\) normalized by the integral in the denominator of Eq. (5) is the probability that a given hydrometeor reaching the surface has fallen from a height between \(z\) and \(z + dz\).

The last expression in Eq. (5) is obtained by integrating the first expression by parts and taking into account that the upward flux of air \(wN\) is approximately independent of \(z\); that is, \(\alpha(Nw)/\alpha z = 0\) (see appendix for details).

We allow for the incompleteness of condensation, which we can define as \(\zeta = \gamma^s(z_2)/\gamma^s(z_1)\), which describes the share of water vapor that has not condensed when the saturated air parcel rising from \(z_1\) has reached \(z_2\). If \(\zeta = 0\), the moist adiabatic \(H_p\) becomes

\[
H_p = z_1 + \int_{z_1}^{z_2} [\gamma^s(z)/\gamma^s(z_1)]dz.
\]

3. Factors influencing \(H_p\)

The advantage of Eq. (5) is that \(H_p\) can be estimated from theory. When the vertical distribution of water vapor follows the moist adiabat, \(H_p\) is unambiguously determined by the surface temperature (see appendix). In Fig. 1a we show the dependence of moist adiabatic \(H_p\) on \(T_s\) for \(R_H = 100\%\) and three values of \(\zeta = \gamma^s(z_2)/\gamma^s(z_1)\): 0, 1/2, and 2/3. We can see that \(H_p\) grows with increasing \(T_s\); for example, for \(\zeta = 0\) we have \(H_p = 3.5\) km at 290 K and 5.3 km at 300 K; that is, \(H_p\) increases by around 50% for a 10-K rise in surface temperature. We also find that \(H_p\) decreases sharply with increasing \(\zeta\); at 300 K and \(\zeta = 1/2\) we have \(H_p = 2.4\) km; Fig. 1a (i.e., \(H_p\) decreases more than twofold compared to the case of complete condensation \(\zeta = 0\)). Height \(z_2\) at \(\zeta = 1/2\) and \(T_s = 300\) K is equal to 4.9 km; Fig. 1a.

In the real atmosphere the lower layer \(z < z_1\) is generally undersaturated with a global mean relative humidity at the surface of about \(R_H = 80\%\). The value of \(z_1\) depends on the temperature lapse rate in the lower atmosphere \(\Gamma_1 = -\alpha T/dz\) for \(z < z_1\) (see appendix). In Figs. 1b–d values of \(z_1\) and a moist adiabatic \(H_p\) for \(\zeta = 0\) are given for \(R_H = 80\%, 60\%,\) and \(40\%,\) respectively, as dependent on surface temperature for three representative values of \(\Gamma_1: 5, 6.5,\) and 9.8 K km\(^{-1}\).

We find that the existence of the undersaturated layer \(z < z_1\) does not change \(H_p\) by much when compared to the case of \(R_H = 100\%\). For example, if \(T_s = 300\) K we have \(z_1 = 0\) km and \(H_p = 5.3\) km for \(R_H = 100\%\) (Fig. 1a), while for \(R_H = 80\%\) we have (i) \(z_1 = 1.2\) km and \(H_p = 5.6\) km with \(\Gamma_1 = 5\) K km\(^{-1}\) and (ii) \(z_1 = 0.46\) km and \(H_p = 5.0\) km with \(\Gamma_1 = 9.8\) K km\(^{-1}\) (Fig. 1b). The value of \(z_1\) grows with decreasing relative humidity. At \(R_H = 40\%, T_s = 300\) K, and \(\Gamma_1 = 5\) K km\(^{-1}\) we have \(z_1 = 4.4\) km and \(H_p = 7.3\) km (Fig. 1d). However, under conditions of low relative humidity, the temperature lapse rate cannot be much smaller than the dry adiabatic lapse rate, so the estimate of \(H_p = 7.3\) km is not realistic. In the realistic case of \(\Gamma_1 = 9.8\) K km\(^{-1}\) at \(R_H = 40\%\) we have \(z_1 = 1.8\) km and \(H_p = 4.8\) km. In other words, despite the condensation zone being elevated by \(z_1\) when compared to the case of 100% relative humidity, the mean height of condensation \(H_p\) does not rise by the same magnitude. This is caused by the temperature dependence of \(H_p\): for \(z_1 \approx 1\) km, temperature \(T_1\) at \(z_1\) is several kelvins lower than at the surface, \(T_1 < T_s\), such that the zone of intense condensation is compressed into a smaller vertical space than it would be if \(T_1 = T_s\). So the resulting \(H_p\) can be even lower than in the case of \(z_1 = 0\).

We can now evaluate \(W_t\) from Eq. (2) at stated values of \(H_p\) and \(P_t\) to see if our assumption \(W_t \gg |w|\) is realistic. Equating Eqs. (2) and (3) we obtain \(W_t = P_t H_p/C,\) where
$C = \int_{0}^{\infty} \rho_c \, dz$ is the amount of condensate and $W_t$ is the mean vertical velocity of condensate in the column. We take the mean precipitation in the tropical region between 30°S and 30°N to be $P_t = 1.3 \, \text{myr}^{-1}$ according to the data of Legates and Willmott (1990), a conservative (i.e., low) value of $H_P = 3.8 \, \text{km}$ (this corresponds to $T_s = 300 \, \text{K}$, $R_H = 80\%$, and $\Gamma = 6.5 \, \text{K km}^{-1}$ for $z > 0$; curve 7 in Fig. 1b), and $C = 10^{-1} \, \text{kg m}^{-2}$ (Wood et al. 2002; O’Dell et al. 2008). Using these values we obtain $W_t = 1.6 \, \text{ms}^{-1}$. This value is about two orders of magnitude larger than typical time-averaged large-scale vertical air velocities ($w < 1 \, \text{cm s}^{-1}$) observed in the tropics (e.g., Rex 1958). We conclude that our assumption $W_t \gg |w|$ is reasonable.

We now consider the case of small terminal velocities, for which Eq. (3) does not hold. Terminal velocity depends on the size of hydrometeors and approaches zero when the condensate particles become vanishingly small. The limiting case is the so-called reversible adiabat, which corresponds to $W_t = 0$ when all condensate travels together with the air and fully evaporates in the region where the moist air descends. Note that surface precipitation in this case is zero so Eq. (3) is inapplicable.

To travel with the air and to reach heights that are significantly larger than $H_P$, the condensate must have a vertical velocity comparable to that of air: $W_t \approx |w|$ and $W_t \ll |w|$. For $C \approx 10^{-1} \, \text{kg m}^{-2}$ and $W_t \ll |w| < 10^{-2} \, \text{ms}^{-1}$ in the tropics, we obtain from Eq. (2) $D < 10^{-2} \, \text{W m}^{-2}$. In other words, even if all condensate in the tropical atmosphere consisted of the smallest condensate particles, their contribution to dissipation rate would not have exceeded $10^{-2} \, \text{W m}^{-2}$. This means that in the tropics the small amount of condensed water that is brought by air updrafts to large altitudes significantly exceeding $H_P$ [Eq. (5)] makes little contribution to $D$ [Eqs. (3) and (5)], with the latter being on the order of $1 \, \text{W m}^{-2}$.

4. Incomplete condensation and precipitation efficiency

As saturated moist air rises, it can mix with drier air from the surroundings. This means that relative vapor
content drops not because of condensation, but because of dilution: water vapor is replaced by dry air in the updraft through turbulent mixing. In such cases condensation is incomplete: the water vapor removed from the ascending air by turbulence has not condensed. This will affect the value of the so-called precipitation efficiency $\varepsilon$. In empirical studies this measure is defined as the ratio of $P_s$ to the inflow of moisture into the updraft. It is commonly observed to be in the vicinity of 20%–50% (Fankhauser 1988).

A common interpretation of these low $\varepsilon$ values from observation is that they reflect reevaporation of condensed water in downdrafts (Newton 1966; Foote and Fankhauser 1973). This presumes that all the water vapor that has flown into the updraft will condense. Since condensation occurs when the relative humidity is equal to unity, this logic would imply that the relative humidity within the updraft should remain high up to the layer where the water vapor mixing ratio has dropped to a negligible value compared to its value at the surface. For example, for a moist adiabat at $T_s = 300$ K, a hundredfold reduction of water vapor mixing ratio corresponds to a height of about 14 km (Fig. 2a). In reality, however, relative humidity drops abruptly much earlier—for example, in hurricanes and their ambient environment it decreases sharply from over 80% to 50%–60% at a height of about 4–5 km (Sheets 1969; Lord and Franklin 1990). The updraft of air lifting the smallest droplets can continue beyond that height, but low relative humidity means that intense condensation cannot.

In other words, precipitation efficiencies less than 1 do not necessarily imply reevaporation of condensed moisture: rather, they can indicate incomplete condensation. Ignoring evaporation, from a simple mass balance consideration we have $(1 - \zeta) \rho_{av} w_s = P_s$, where $\rho_{av}$ is the density of water vapor and $w_s$ is the vertical air velocity at the cloud base and $\zeta = \gamma(z_2)/\gamma(z_1) \sim r^\theta(z_2)/r^\theta(z_1)$ is equal to the ratio of the water vapor mixing ratio at height $z_2$ where condensation discontinues and its value at the cloud base $z = z_1$. The flux of water vapor flowing into the cloud is given by $\rho_{av} w_s$. In this case $\varepsilon = P_s/(\rho_{av} w_s) = 1 - \zeta$. If $\varepsilon = 1/3$, this means $\zeta = 2/3$; that is,
the condensation zone reaches upward to a height where the water vapor mixing ratio decreases by one-third as compared to its value at the cloud base. As we show in Fig. 1a, high values of $\zeta$ and, hence, low precipitation efficiencies $\varepsilon$ are associated with relatively low $H_P$.

### 5. Comparison with the results of Pauluis et al.

#### a. The scale of $H_P$

Pauluis et al. (2000) based their estimate of $D$, the first of this kind in the meteorological literature (Rennó 2001; Pauluis et al. 2001), on Eq. (3). They noted that, if reevaporation is neglected, $H_P$ is equal to the average height where condensation occurs. This is correct, but some of the subsequent assumptions and derivations are not as well justified. Equation (5) of Pauluis et al. (2000) defines $H_P$ as $H_P = \int_0^z r^*\,dz$, where $r^*$ is saturated water vapor mixing ratio. This equation misses the normalization factor $r_c = r(0)$ in the denominator [cf. our Eq. (6) with $z_1 = 0$]. Turning to quantitative estimates, Pauluis et al. (2000) proposed that the scale height of the saturated water vapor mixing ratio in the tropics is about 2.5–3 km. Pauluis et al. (2000) cite Emanuel and Bister (1996), who provide a “scale height for water vapor” of around 3 km. However, the scale height of saturated water vapor and the scale height of its mixing ratio are different atmospheric characteristics. They depend differently on temperature and, hence, height. In the tropical troposphere, for example, they range from 0 to 5 km and to 10 km for the scale height of water vapor and its mixing ratio, respectively (e.g., Makarieva and Gorshkov 2010, their Fig. 1).

Following their initial suggestions, Pauluis et al. (2000) offered several arguments as to why $H_P$ in Eq. (3) should be several times higher than the scale height of the saturated water vapor mixing ratio and takes a value of 5–10 km. First, they noted that the real $H_P$ is greater than condensation height obtained from a moist adiabat because there is an undersaturated region in the subcloud layer. As we discussed in section 3, while real, the effect is small and does not necessarily lead to an increase in $H_P$.

Second, Pauluis et al. (2000) proposed that an increase in $H_P$ can be induced by the entrainment of the unsaturated air parcels into the region of saturated ascent. We disagree inasmuch as the entrainment of dry air would cause the temperature lapse rate to rise above the moist adiabatic value, so the temperature would drop more rapidly with height. If, despite the dry air entrainment, the condensation continued in the updraft prompted by this additional cooling in the ambient environment, the change in local lapse rate will reduce rather than raise the mean $H_P$. Curve 7 in Fig. 1b illustrates the dependence of $H_P$ on $T_c$ for $R_H = 80\%$ and a mean tropospheric lapse rate of 6.5 K km$^{-1}$ instead of moist adiabatic lapse rate. In this case $H_P$ depends little on surface temperature and ranges between 3 and 4 km. If, on the other hand, condensation is discontinued by the decline in relative humidity associated with the removal of water vapor and its replacement by dry air in the region of ascent, then $H_P$ is limited by the height where the dry air entrainment occurred. In neither case does $H_P$ increase.

#### b. Uplift of condensate particles

Pauluis et al. (2000) also mention that the real precipitation pathlength is increased by the fact that some hydrometeors are lifted by updrafts to high altitudes. O. Pauluis (2013, personal communication) suggested that condensate can be uplifted in the form of small particles that then grow and precipitate from a height that significantly exceeds $H_P$ [Eq. (5)].

The uplift of small condensate particles, while certainly happening, is unlikely to cause a significant increase in the mean precipitation height. Indeed, consider the limiting case when all condensate is formed as tiny particles, which can be generally defined as particles that have a negligible velocity relative to the air. This fraction of condensate is sometimes referred to as cloud (or airborne) water to differentiate it from precipitating water comprising bigger particles that are falling down at an appreciable velocity (Ooyama 2001; Jiang et al. 2008). Let us assume that all of these tiny particles are brought upward by the updraft up to a certain height $z_g$, where they suddenly grow big and rapidly precipitate to the ground. In such a case at any height the condensate is represented by cloud water with density $\rho_{cc}$ (tiny particles flowing up with the air) and precipitating water (big particles falling down). Since all tiny particles travel with the same air parcel where they were formed, the mixing ratio of total water in the parcel (excluding precipitating water) should then remain constant: the mixing ratio of saturated water vapor diminishes, while the mixing ratio of cloud water grows with height. Such cloud water content distribution is referred to as adiabatic and can be calculated from the vertical profile of $\gamma^a(z)$; see Fig. 3a and Eq. (A11) in the appendix.

The total mass of tiny condensate particles in the column below $z_g$, sometimes referred to as cloud water path $\sigma = \int_0^{z_g} \rho_{cc}(z)\,dz$, is shown in Fig. 3b. We note that cloud water path grows very rapidly with $z_g$. If all condensate is taken to $z_g = z_{max} = 20$ km, the total amount of tiny particles in the column would be $\sigma = 90$ kg m$^{-2}$.

Therefore, the observed $\sigma$ helps us judge the mean height of the uplift above the condensation layer. While discrimination of cloud water from rainwater represents a considerable challenge, it is well established that for
mild rains cloud water constitutes most part of the total column water, while during strong rain events this proportion declines in favor of precipitating water (O’Dell et al. 2008). As an example, in hurricanes cloud water represents 5%–20% of total condensate in the column (Jiang et al. 2008). In deep convective clouds precipitating liquid water and ice contribute nearly equally to total water in the column at around 2 kg m⁻² each (Liu and Curry 1998; Jiang et al. 2008). Therefore, a value of $\sigma_o = 0.5$ kg m⁻² reasonably represents observations in both deep convective clouds [corresponding to 10% of total water content in hurricanes (Jiang et al. 2008)] as well as the shallow cumuli. In shallow cumuli such values are common to the early stages of precipitation events when $\sigma$ is at its maximum (e.g., Bennartz et al. 2010).

We see from Fig. 3b that $\sigma = \sigma_o = 0.5$ kg m⁻² corresponds to $z_g = 1.4$ km (i.e., when the uplift distance is $z_g - z_l = 0.6$ km) or just 11% of $H_p = 5.3$ km calculated without uplift. The potential influence of the uplift on $H_p$ estimated from Eq. (5) thus appears minor. For example, if the uplift continued up to 10 km, we would observe $\sigma = 50$ kg m⁻² $\gg \sigma_o$, which, as discussed above, strongly disagrees with observations. The fact that the uplift of condensate from the cloud base (where condensation rate is the highest) is insignificant is underlined by observations of the vertical profiles of $\rho_{cc}$ in individual clouds. Indeed, the adiabatic $\rho_{cc}$ grows with $z$ as shown in Fig. 3a. Such a growth in real clouds in different regions of the world occurs over no more than several hundred meters beyond which there is a sharp decline of $\rho_{cc}$ (Das et al. 2010; Bennartz et al. 2010; Earle et al. 2011).

We know from observations that condensation in updrafts continues well above $z_g \sim 2$ km. The condition $\sigma \ll \sigma_o$ means that only a certain part ($x < 1$) of the condensate can be uplifted to a significant height. In Figs. 3c and 3d two scenarios of condensate uplift are shown: both yield similar results; see appendix for details. In Fig. 3c a certain share $x$ of condensate formed at each height is lifted to height $z_{\max} = 20$ km, while the remaining condensate $(1 - x)$ instantaneously
precipitates to the ground from the height where it was formed and makes no contribution to cloud water path.

The dependence of mean height of precipitation $H_s = z_{\text{max}} + (1 - x)H_p$, where $H_p$ is calculated from Eq. (5), is shown together with the resulting cloud water path $\sigma_s$. For $\sigma_s = \sigma_o = 0.5 \text{ kg m}^{-2}$ we have $x = 0.005$ and $H_s = 5.36 \text{ km}$. Thus, in order to have $0.5 \text{ kg m}^{-2}$ of cloud water in this scenario we would need to uplift only 0.5% of total condensate to $z_{\text{max}} = 20 \text{ km}$. This would not increase the mean $H_p = 5.3 \text{ km}$ by more than 1% and is thus negligible. In Fig. 3d it is assumed that below a certain height $z_s$ all condensate instantaneously precipitates from where it was formed, while from $z > z_s$ all condensate is taken up to $z_{\text{max}} = 20 \text{ km}$. Again, the resulting mean $H_s$ and $\sigma_s$ are shown as functions of $z_s$. One can see that the realistic value of $\sigma_s = \sigma_o = 0.5 \text{ kg m}^{-2}$ corresponds to $H_s$ values indistinguishable in practice from $H_p = 5.3 \text{ km}$ obtained assuming zero condensate uplift. For $\sigma_s = \sigma_o = 0.5 \text{ kg m}^{-2}$, we have $z_s = 12 \text{ km}$ and $H_s = 5.42 \text{ km}$.

We emphasize that $\sigma$ is the amount of cloud water observed in actual updrafts. The climatological mean value of cloud water path in the tropics is about 4 times less than our reference value $\sigma_o = 0.5 \text{ kg m}^{-2}$ ($\text{O’Dell et al. 2008}$). As there is some horizontal transport of condensate from the updraft to the ambient environment, the actual amount of condensate uplifted to a large height can be larger than calculated from Figs. 3b–d for $\sigma_o = 0.5 \text{ kg m}^{-2}$. However, since this horizontal export of condensate from the updraft does not normally exceed 50% of total condensate formed in the updraft (Leary and Houze 1980; Jiang et al. 2008), it does not change our conclusion about the insignificant impact of uplift on the mean $H_p$. Repeating all the above calculations for $\sigma_s = 2\sigma_o = 1 \text{ kg m}^{-2}$ produces $z_s - z_1 = 1 \text{ km}$ in Fig. 3b, $x = 0.01$ and $H_s = 5.44 \text{ km}$ in Fig. 3c, and $z_s = 11.2 \text{ km}$ and $H_s = 5.56 \text{ km}$ in Fig. 3d. We thus conclude that the observed values of cloud water during real precipitation events are inconsistent with a significant part of condensate being lifted by updrafts significantly above the height where condensation occurred. At most this effect can lead to an approximately 10% increase of precipitation height as estimated from Eq. (5).

**c. Reevaporation of condensate**

The final argument put forward by Pauluis et al. (2000), and the only quantitative one, concerns reevaporation. They presume that this effect can lead to a significant underestimate of the real value of $H_p$. Pauluis et al. (2000) cite the work of Fankhauser (1988) and Ferrier et al. (1996) to support the statement that a significant part (from half to two-thirds) of all condensed moisture actually reevaporates and does not hit the ground. From this, Pauluis et al. (2000) suggest that if evaporation occurs uniformly as the hydrometeors are falling, this process increases the effective precipitation pathlength by a factor of 1.5–2. We note that Fankhauser (1988), who investigated empirical data on the water budget of convective clouds including precipitation efficiency, does not mention any quantitative estimate of the reevaporation of condensed moisture. Ferrier et al. (1996, p. 2105), on the other hand, do report the magnitude of reevaporation as compared to total condensation within a squall, but their results come from a numerical model rather than observational evidence. To estimate the actual rate of evaporation of condensate in the downdrafts we require accurate estimate of condensate transport within the cloud—rather than measuring the transport of total moisture that is dominated by water vapor. Estimates of sufficient accuracy are not available. We can, in contrast, be confident that the effect of incomplete condensation associated with low precipitation efficiency is real. But, as discussed in the previous section, this will decrease rather than increase the estimate of $H_p$.

Furthermore, even if evaporation in downdrafts did constitute a significant fraction of total condensation, we would argue that the suggestion of Pauluis et al. (2000) about a factor of 1.5–2 increase in effective $H_p$ would still be incorrect. Let us first consider how this conclusion was reached. The argument of Pauluis et al. (2000) appears to derive from Eq. (3) along with the one-dimensional continuity equation for condensate particles. If $j_P = \rho_c(z)W_t$ is the downward flux of condensate at point $z$, then the continuity equation is $\partial j_P/\partial z = E$, where $E > 0$ is evaporation. If, following Pauluis et al. (2000), we assume that $E$ is constant and that $j_P(0) = (1/3)j_P(H_P)$ (evaporation has decreased the original precipitation flux by two-thirds as it traveled from $z = H_P$ to $z = 0$), then we have $j_P(z) = j_P(0)(1 + 2z/H_P)$. This allows us to calculate $D$ from Eq. (2) as $D = \int_0^{H_P} j_P g H_P dz = 2 j_P(0) g H_P = 2P_g g H_P$. If this logic were correct, we could conclude, as did Pauluis et al. (2000), that Eq. (3) indeed underestimates the actual dissipation by half.

There are, however, two debatable premises in this reasoning. The first is neglecting that dissipation rate $\rho_c W_t$ and evaporation behave differently with respect to droplet size. The smallest droplets make a larger contribution to evaporation and a smaller one to dissipation. A brief illustration follows. Suppose that absolute evaporation rate is proportional to droplet area $s \propto d^2$, where $d$ is droplet diameter. If we have equal amounts $M$ (g) of small droplets with diameter $d_1$ and large droplets with diameter $d_2 > d_1$, the rate of depletion of total condensate due to the evaporation from small droplets will be $d_2/d_1$ times faster than from large droplets [evaporation rate $\propto Ns \propto (M/m)d^2 \propto M/d$, where $N$ is the number of droplets, $s$ is the area of a droplet, $W_t$ is the total mass flux, and $M$ is the total mass of condensate.]
where \( s \approx d^2 \) is droplet’s surface area, \( N \approx M l m \approx M d^3 \) is the number of droplets, and \( m \approx d^3 \) is droplet mass. In comparison, because terminal velocity grows with increasing droplet diameter, the contribution of the small droplets to total dissipation will be lower than that of the large droplets. In theory, for spherical droplets with \( W_T \propto d^2 \), it will be lower by a factor of \( (d_2/d_1)^2 \). For example, with \( d_2/d_1 = 10, 90\% \) of all evaporation comes from the droplets that make a 1\% contribution to total dissipation. The real situation of the dependence of evaporation and terminal velocity on droplet size is more complex (e.g., Seifert 2008). But the general pattern remains the same: evaporation and dissipation depend differently on droplet size. If, under the assumption that droplets of all sizes evaporate equally, Pauluis et al. (2000) concluded that this led to an underestimate of \( D \) by 1.5–2 times (50%–100%), the account of a larger impact on evaporation by smaller droplets can potentially reduce this estimate to an insignificant magnitude.

The second debatable premise is that it implicitly use \( j_p = \rho_c(z)W_t \) to represent the downward flux of condensate. This neglects the role of vertical air movements in transporting the smallest condensate particles. Rather, we would argue that they should have used \( j_k = \rho_c(z)(w - W_t) \), where \( w < 0 \) is the downward velocity of air. The \( w \) term is particularly important in consideration of evaporation, because the smallest droplets with \( W_t \ll -w \) are so slow that they can be only transported by the downdraft. This means that the continuity equation \( \partial j_p/\partial z = E \) underlying the reasoning of Pauluis et al. (2000) is not valid. The correct equation \( \partial j_k/\partial z = E \) is uninformative as long as the distribution of \( w \) remains unknown.

To summarize, our analysis does not support the claim of Pauluis et al. (2000) that precipitation path-length should be several times higher than the value of \( H_p \) given by Eq. (5).

6. Numerical estimate of \( D \)

We have shown that \( H_p \) grows with decreasing incompleteness of condensation \( \zeta \), increasing surface temperature and decreasing \( \Gamma_1 \) for \( z < z_1 \) (Fig. 1). Among these, \( \zeta \) is both the least known and the most influential (Fig. 1a). It is closely linked to convection depth. We suggest that the uncertainty of the mean \( D \) values for the tropical region is largely determined by the uncertainty in \( H_p \), which itself largely reflects uncertainty of \( \zeta \). Taking \( T_s = 300 \) K as the mean surface temperature during precipitation in the tropical region, \( \Gamma_1 = 6.5 \) K km\(^{-1} \) and \( R_{HI} = 80\% \), we obtain \( H_p = 5.3 \) km for complete condensation \( \zeta = 0 \) (Fig. 1b).

This figure provides an upper limit on the real value of \( H_p \). About one-third of tropical precipitation is yielded by shallow clouds and two-thirds by convective clouds (Tokay and Short 1996; Folkins and Martin 2005), with the tops of shallow cloud not extending beyond 3–5 km and being on average significantly smaller (Miles et al. 2000; Folkins and Martin 2005). The available data on vertical profiles of relative humidity during intense precipitation events also indicate that the condensation layer (i.e., the layer where relative humidity is sufficiently high to allow for condensation) does not generally exceed 4–5 km (Sheets 1969; Folkins and Martin 2005; Lord and Franklin 1990). This suggests that typically about half of all the water vapor does not condense in convective updrafts (\( \zeta \sim 0.5 \)). Likewise, the characteristic values of \( \zeta \) reported for individual thunderstorms range between 20% and 50% (Fankhauser 1988), which, as we discussed above, correspond to \( \zeta \) from 0.8 to 0.5. For \( T_s = 300 \) K, \( \Gamma_1 = 6.5 \) K km\(^{-1} \), and \( R_{HI} = 80\% \) and the upper condensation height \( z_2 = 5 \) km, we have \( \zeta = 0.5 \) and \( H_p = 2.8 \) km. This value provides a lower estimate of \( H_p \).

We note that this lower estimate corresponds to a below-average incompleteness of condensation (\( \zeta = 1/2 \)). Therefore, the nearly twofold difference between the lower and upper limit of \( H_p \) (the latter calculated for complete condensation) potentially harbors all of the above discussed minor impacts that could drive \( H_p \) upward by a magnitude of order 10%, including reevaporation of droplets and condensate uplift. This supports our proposal that \( H_p \sim 5 \) km is a robust upper limit to precipitation height in the tropical atmosphere.

Pauluis et al. (2000) estimated \( D \) from the mean latent heat flux \( \mathcal{L} \) instead of precipitation \( P_s = \mathcal{L}/L_v \) in the tropics considering that \( D/\mathcal{L} = gH_p/L_v \), where \( L_v = 2.5 \times 10^6 \) J kg\(^{-1} \) is the heat of vaporization. Using \( \mathcal{L} = 100 \) W m\(^{-2} \) and \( H_p \) values of 2.8 and 5.3 km, we obtain a range of 1.1–2.1 W m\(^{-2} \) for the mean tropical value of \( D \). For the global mean temperature \( T_s = 288 \) K at \( R_{HI} = 80\% \) and \( \Gamma_1 = 6.5 \) K km\(^{-1} \), we have \( H_p = 3.6 \) km for \( \zeta = 0 \) (Fig. 1b) and \( H_p = 1.5 \) km for \( \zeta = 2/3 \). Using the mean value of 2.5 km and mean global precipitation of 1 m yr\(^{-1} \) (L’vovitch 1979), we obtain a global mean value of \( D \sim 0.78 \) W m\(^{-2} \) from Eq. (3). This means \( 4 \times 10^{14} \) W for Earth as a whole and \( 1.2 \times 10^{14} \) W for the gravitational power of precipitation on land. [The estimate of \( 10^{14} \) W for land was previously obtained based on Eq. (3) in a different context discussing renewable energy sources (Gorshkov 1982, p. 6).]

If the vertical profile of precipitation \( P(z) \) is known, the value of \( D \) can be estimated directly from Eq. (2) under the assumption that \( P(z) = \rho_c(z)W_t \). This was recently done by Pauluis and Dias (2012), who used satellite-derived \( P(z) \) profiles from the Tropical Rainfall Measurement Mission and estimated \( D \) for the
tropical region between 30°S and 30°N to be 1.5 W m$^{-2}$. Using this value Pauluis and Dias (2012) estimated $H_P$ from Eq. (3) by dividing $D$ [Eq. (2)] by $P_{s g}$, $H_P = D/(P_{s g})$. They obtained $H_P$ estimates of 4.9 km for the ocean and 5.8 km for land and proposed that the difference indicates more intense convection over land than over the ocean.

However, the derivation of $H_P$ from Eq. (3) is potentially misleading, as such an estimate is sensitive to the estimate of surface precipitation. If surface precipitation is underestimated in the lowest kilometer, this may make little impact on the column-integrated $D$ [Eq. (2)] but will have a considerable impact on the value of $H_P$. According to the revised Fig. 2 of Pauluis and Dias (2012), precipitation in the lowest 1 km makes about a 1/5 contribution to the column-integrated value of $D$. If, as discussed by Pauluis and Dias (2012), precipitation in this lowest region is underestimated by 25% (Durden et al. 1998; Bowman 2005) and constitutes 75% of the real value, this corresponds to a 25%/5 = 5% underestimate in total $D$. But $H_P = D/gP_s$ is then overestimated by a factor of (100% – 5%)/75% = 1.3. Applying this correction factor to the $H_P$ estimates of Pauluis and Dias (2012) we obtain $H_P = 4.9/1.3 = 3.8$ km instead of 4.9 km for the ocean and $H_P = 5.8/1.3 = 4.5$ km instead of 5.8 km for land. Both values are within the 3–5-km range estimated from our theoretical analysis. They are close to $H_P = 3.8$ km obtained for $R_H = 80\%$, $T_s = 300$ K, and mean tropospheric lapse rate 6.5 K km$^{-1}$ in a saturated atmosphere; see curve 7 in Fig. 1b.

Tropical land includes wet regions like the Amazon and Congo forests, where precipitation is 2–3 times higher than over the nearby ocean (Makarieva et al. 2013a). But the tropics also include very dry regions such as the Sahara Desert and the Australian interior. Combining these wet and dry regions under one category and calculating a single mean precipitation profile for all tropical land is of questionable value given the diversity of physical settings. (Note that these concerns have less significance for a tropicwide estimate—the values of which are dominated by the oceans.) In the driest regions of Earth where surface precipitation tends to zero, estimating $H_P$ from surface precipitation lacks any physical meaning, as $H_P \rightarrow \infty$ at $P_s \rightarrow 0$. Thus, any estimated value for $H_P$ does not carry information about the vertical distribution or intensity of precipitation. This may explain why some of the highest $H_P$ values obtained by Pauluis and Dias (2012) occur far inside the Sahara Desert, where convection is usually absent.

### 7. Frictional dissipation and atmospheric circulation

We now consider how the frictional dissipation relates to the dynamic power of atmospheric circulation. Upon condensation, two types of potential energy are formed: first, the potential energy of falling hydrometeors as dictated by $H_P$ [Eq. (5)], and second, the potential energy of the nonequilibrium pressure gradient that results from the disappearance of water vapor from the gas phase. The key peculiarity of this second process is that this potential energy is coupled to the vertical motion of moist air and is released only when moist air moves upward. Previously, we have argued that the dynamic power of condensation-induced circulation per unit surface area (W m$^{-2}$) can be estimated as $Q = P_sRT/M_w$, where $R = 8.31$ J mol$^{-1}$ K$^{-1}$ is the universal gas constant and $T$ is the mean temperature in the atmospheric layer where condensation occurs (Makarieva and Gorshkov 2010, 2011; Gorshkov et al. 2012; Makarieva et al. 2013b).

For a global mean value of $P_s$, 1 m yr$^{-1}$ (L’vovich 1979; Legates and Willmott 1990) and $T = 270$ K, we have $Q = 4$ W m$^{-2}$. For the tropical mean $P_s = 1.3$ m yr$^{-1}$, we have $Q = 5.2$ W m$^{-2}$. Since under conditions of hydrostatic equilibrium the work done by the vertical pressure gradient is compensated by the work done by gravity, the kinetic energy of the large-scale airflow derives from the horizontal pressure gradient alone. The power of this horizontal force per unit air volume is equal to $uV_P$, where $u$ is the horizontal velocity. The dynamic power of atmospheric circulation (the rate at which the kinetic energy is generated) can therefore be estimated from the observed horizontal pressure gradients and the observed $u$ values. It can also be estimated as the power of turbulent dissipation of the airflow under the assumption that in the stationary case the dynamic power that creates the kinetic energy is equal to the power of turbulent dissipation $D_t$ of this energy. The available global-mean observation-based estimates of these powers are in the range of 2–4 W m$^{-2}$ (Oort 1964; Lorenz 1967; Peixoto and Oort 1992; Marvel et al. 2013). Our global-mean estimate, derived from basic principles, falls near the upper edge of this range.

The ratio of the powers of the two processes—the frictional dissipation power of hydrometeors and the dynamic power of condensation-induced circulation—is given by

$$\frac{D}{Q} = \frac{H_P}{h_v'}, \text{ where } h_v' = \frac{RT}{M_w g}.$$
This ratio does not depend on $P_s$ but grows with temperature owing to the dependence of $H_P$ on $T_s$ (Fig. 1).

The value of $h_o$ [Eq. (7)] has the meaning of the scale height of (unsaturated) water vapor in hydrostatic equilibrium; at $T_s = 300\,K$ it is equal to 14 km. With the large-scale values of $H_P$ not exceeding 6 km at mean surface temperatures not exceeding 300 K (Fig. 1), we obtain a general estimate of $D/Q < 0.4$.

We can estimate when $D$ exceeds $Q$ by noting that $D$ grows with surface temperature (Fig. 1). At $R_H = 80\%$ and $\Gamma_1 = 5\,K\,km^{-1}$, $D$ equals $Q$ at $T_s = 323\,K$ (50°C) (Fig. 1b). At higher temperatures in a moist adiabatic atmosphere, any significant circulation due to condensation will be prevented because of the insufficient dynamic power to overcome the energy losses associated with frictional dissipation due to precipitation. If the atmosphere is not moist adiabatic but has a constant lapse rate of $\Gamma = 6.5\,K\,km^{-1}$, $D$ grows much more slowly with increasing surface temperature (see Fig. 1, curves 7). It does not approach $Q$ anywhere at $T_s < 360\,K$ [i.e., in the entire range where the approximation $\gamma \ll 1$ on which Eq. (5) is based holds]. We thus conclude that frictional dissipation due to precipitation is insufficient to arrest condensation-induced atmospheric circulation on Earth.

8. Discussion

To investigate how an atmospheric phenomenon responds to changes in atmospheric parameters, it is important to establish a sound theoretical basis concerning the key physical relationships. In this paper, building from basic physical principles and relationships, we evaluated the rate of frictional dissipation associated with precipitation. We discussed how precipitation pathlength $H_P$ is the key parameter controlling this rate and investigated how it depends on surface temperature, humidity, and the vertical extent of the area where condensation occurs.

The concordance between our results and those of Pauluis et al. (2000) justify a summary. Pauluis et al. (2000) proposed that the scale of $H_P$ values is set by condensation height, which they estimated as the mean scale height of the water vapor mixing ratio in a saturated atmosphere. This is correct and corresponds to our Eq. (6) with $z_1 = 0$ and $z_2 = \infty$, but their suggestion of the characteristic condensation height values in the tropics (2.5–3 km) is in our view an underestimate. The moist adiabatic condensation height in the tropics ranges between 3.5 and 5 km depending on temperature.

On the other hand, Pauluis et al. (2000) proposed several processes and factors that elevate $H_P$ by several times up to $H_P \sim 5–10\,km$. These factors and processes were (i) the existence of the undersaturated layer at Earth’s surface, (ii) uplift of condensate particles to large heights by convective currents, and (iii) reevaporation of condensate particles. Here we argued (see section 5) that none of these have a major influence on the value of $H_P$. At the same time, we showed evidence that the incompleteness of condensation $\zeta$, a factor not considered by Pauluis et al. (2000), is highly influential.

The net result of these analyses was that $H_P$ in the tropics is likely to be between 2.8 and 5.3 km (section 6)—that is, about half the value (5–10 km) suggested by Pauluis et al. (2000). It is important to emphasize that the theoretical analysis pertains the determination of $H_P$ only, while dissipation rate $D$ [Eq. (3)] depends on the product of $H_P$ and surface precipitation $P_s$. Using the same value of $P_s$, as Pauluis et al. (2000) produces a range of tropical $D$ from 1.1 to 2.1 W m$^{-2}$ substantially below the range 2–4 W m$^{-2}$ proposed by Pauluis et al. (2000).

Satellite-derived vertical profiles of precipitation provide an independent source of information on $D$ that can then be estimated from Eq. (2). The analysis of satellite data by Pauluis and Dias (2012) yielded $D = 1.5\,W\,m^{-2}$, which fits within the theoretical range established in this paper. Pauluis and Dias (2012) argued that as the satellite data they used tended to underestimate surface precipitation by about 25%, this $D$ value is likely to be a lower limit. We, however, showed that such a significant underestimate of surface precipitation results in only a minor (about 5%) underestimate of $D$. Note also that precipitation rates in the upper part of the atmosphere can be, on the contrary, overestimated by the radar (Durden et al. 1998). Until the relevant uncertainty estimates are explicitly taken into consideration, the satellite-derived estimates cannot constrain $D$ any more accurately than do our theoretical estimates.

An important result is the growth of $D$ with surface temperature $T_s$ owing to growth in $H_P$ (Fig. 1). We showed that at $T_s \sim 320\,K$ the frictional dissipation associated with precipitation equals the total observed power of global circulation provided the latter is driven by condensation-induced dynamics (section 7). This effect has profound implications for climatic stability: as circulation slows, it will affect the vertical lapse rate, oceanic circulation, and horizontal heat fluxes, all of which are significant determinants of global conditions. Further theoretical and empirical studies of this process are needed.

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APPENDIX

Equations for Calculating $z_1$ and $H_P$

Equation (5) is obtained using Eq. (4) and considering that

$$\int_{z_1}^{z_2} S(z) \, dz = wN \gamma^* z_1^2 + \int_{z_1}^{z_2} wN \gamma^* \, dz + \int_{z_1}^{z_2} \frac{\partial(wN)}{\partial z} \gamma^* \, dz$$

$$\simeq wN \left( \gamma^* z_1^2 + \int_{z_1}^{z_2} \gamma^* \, dz \right), \quad (A1)$$

$$\int_{z_1}^{z_2} S(z) \, dz = wN \gamma^* z_1^2 + \int_{z_1}^{z_2} \frac{\partial(wN)}{\partial z} \gamma^* \, dz$$

$$\simeq wN \gamma^* z_1^2. \quad (A2)$$

From the one-dimensional stationary continuity equation we have $\partial(wN)/\partial z = -S$. This means that the terms discarded in Eqs. (A1) and (A2) constitute a small magnitude on the order of $\gamma^* \ll 1$ as compared to the initial terms. The quantity $wN$ changes little with $z$ (by a relative magnitude on the order of $\gamma^*$) as compared to $\gamma^*$ that changes severalfold. Therefore, $wN$ can be assumed to be constant and cancelled from both the denominator and nominator in ratio (5). The inaccuracy of the resulting expression for $H_P$ [Eq. (5)] is on the order of $\gamma^*$ and, for temperatures of interest, does not exceed 10%.

The system of equations for moist adiabat solved to calculate $H_P$ in Fig. 1a is as follows [Makarieva and Gorshkov 2010; Gorshkov et al. 2012, their Eq. (24)]:

$$\frac{1}{T} \frac{\partial T}{\partial z} - \frac{\mu p}{p} \frac{\partial p}{\partial z} + \frac{\mu \xi}{1 - \gamma^*} \frac{\partial \gamma^*}{\partial z} = 0, \quad (A3)$$

$$\frac{1}{\gamma^*} \frac{\partial \gamma^*}{\partial z} - \frac{\xi}{T} \frac{\partial T}{\partial z} + \frac{1}{p} \frac{\partial p}{\partial z} = 0, \quad (A4)$$

$$- \frac{1}{p} \frac{\partial p}{\partial z} \frac{Mg}{RT} = 0, \quad (A5)$$

where temperature $T$, air pressure $p$, and $\gamma^* = p_v^* / p$—the relative partial pressure of saturated water vapor—are functions of height $z$. Here, $\xi = L_v / (RT)$. $L_v = 45 \times 10^3$ J mol$^{-1}$ is heat of vaporization, $R = 8.3 J$ mol$^{-1}$ K$^{-1}$ is the universal gas constant, $M = (1 - \gamma^*) M_d + \gamma^* M_w$, $M_d = 29 g$ mol$^{-1}$, $M_w = 18 g$ mol$^{-1}$, $\mu = R c_p = 2/7$, and $c_p$ is the molar heat capacity of air at constant pressure (J mol$^{-1}$ K$^{-1}$). Equation (A3) results from the first law of thermodynamics for moist air saturated with water vapor. Equation (A4) derives from the definition of $\gamma^*$ combined with the Clausius–Clapeyron law. Equation (A5) is equivalent to the condition of hydrostatic equilibrium [Eq. (1)] for ideal gas.

The boundary conditions for the surface $z = 0$ at a given surface temperature $T_s$ read

$$T = T_s, \quad \text{(A6)}$$

$$p = p_s, \quad \text{(A7)}$$

$$p_v^*(T) = p_v^{*0} \exp(\xi_0 - \xi), \quad \text{(A8)}$$

where $p_v^{*0}$ and $\xi_0 = L_v / (RT_0)$ correspond to some reference temperature $T_0$. We take $T_0 = 303 K$, $p_v^{*0} = 42 hPa$, and the standard value for the atmospheric pressure $p_s = 1013$ hPa. The dependence of $L_v$ on temperature is neglected. In this case, $\xi_0 = 18$.

For a fully saturated atmosphere $z_1 = 0$ in Eq. (5). Numerical evaluation of the system of Eqs. (A3)–(A5) allows us to obtain the unknown functions $T(z)$, $p(z)$, and $\gamma^*(z)$ and to calculate $H_P$ [Eq. (5)] as shown in Fig. 1a.

In Figs. 1b–d the atmosphere at the surface $z = 0$ is not saturated and has a relative humidity of 80%, 60%, and 40%, respectively. To find height $z_1$ where the relative humidity reaches unity, we assume that within the range $0 \leq z \leq z_1$, the nonsaturated $\gamma = p_v / p$ is constant and temperature $T(z)$ drops with height $z$ at a constant lapse rate $\Gamma_1$, $T(z) = T_s - \Gamma_1 z$. Then the nonsaturated pressure $p_v$ of water vapor is given by

$$p_v(z) = p_v^*(T_s) R_H \frac{p(z)}{p_s}, \quad \text{(A9)}$$

where $R_H$ is relative humidity at the surface ($z = 0$), saturated pressure $p_v^*$ of water vapor is governed by the Clausius–Clapeyron law [see Eq. (A8)], and pressure $p(z)$ conforms to the condition of hydrostatic equilibrium [Eq. (A5)] with $M \simeq M_w$. Height $z_1$, where relative humidity becomes unity and condensation commences, is a function of $T_s$. We find height $z_1$ as the solution of equation

$$p_v(z_1) = p_v^*(T_s), \quad T_1 = T(z_1) = T_s - \Gamma_1 z_1. \quad \text{(A10)}$$

The atmosphere is assumed to be saturated and moist adiabatic within the range $z_1 \leq z \leq z_2$. We need to evaluate the system of Eqs. (A3)–(A5) in order to find functions $T(z)$, $p(z)$, and $\gamma^*(z)$. However, the boundary conditions should now be imposed not at the ground surface but at $z = z_1$, so that $T = T_1$ and $p = p(z_1)$ instead of $T_s$ [Eq. (A6)] and $p_v$ [Eq. (A7)] at $z = 0$. Using the obtained solutions, precipitation height is calculated from Eq. (5). We note that an equation for moist
adiabat that accounts for the horizontal pressure gradient [Gorshkov et al. 2012, their Eq. (36)] produces slightly smaller heights $H_P$ than Eq. (A3); e.g., for $T_s = 300\, K$, $\Gamma_1 = 6.5\, K\, km^{-1}$, and $R_{H_P} = 80\%$, we have $H_P = 5.1\, km$ instead of $H_P = 5.3\, km$ yielded by Eq. (A3).

Curves 7 in Figs. 1b–d are obtained for an atmosphere that is unsaturated for $z < z_1$, saturated for $z \approx z_1$, and has a constant temperature lapse rate $\Gamma = 6.5\, K\, km^{-1}$ for $z \approx 0$. The solution for $z \approx z_1$ is obtained by solving the system of Eqs. (A4) and (A5) and $\partial T/\partial z = -\Gamma$ instead of Eq. (A3).

Having obtained the dependencies $T(z)$, $p(z)$, and $\gamma^a(z)$, one can calculate the adiabatic cloud water content $\rho_{cc}$ (kg m$^{-3}$), Fig. 3a, as follows:

$$\rho_{cc}(z) = [\gamma^a(z_1) - \gamma^a(z)]N(z)M_v. \quad \text{(A11)}$$

Here, $\gamma(z) = \gamma_c$ for $0 \leq z < z_1$. The air molar density $N(z)$ satisfies the equation of state for ideal gas $p = NRT$ at point $z$. The cloud water path $\sigma$ (kg m$^{-2}$), Fig. 3b, is defined as follows:

$$\sigma = \int_{z_1}^{z_g} \rho_{cc}(z) \, dz, \quad \text{(A12)}$$

where the upper limit $z_g$ of integration runs within the range $z_1 \leq z_g \leq z_{\text{max}}$. For all examples considered, we choose $z_{\text{max}} = 20\, km$. The value of $\sigma$ describes the amount of cloud water in the atmospheric column below $z_g$.

In Fig. 3c we assume that a share $x$ of condensate formed at a given height is lifted by updrafts up to $z_{\text{max}}$, while the remaining part $(1-x)$ of condensate precipitates instantaneously to the ground from where it was formed. Then the mean precipitation height $H_x$ becomes

$$H_x = (1-x)H_P + xz_{\text{max}}. \quad \text{(A13)}$$

Here, $H_P$ is given by Eq. (5). The mean cloud water path $\sigma_x$ is just $\sigma_x = x\sigma$, where $\sigma$ is calculated according to Eq. (A12) with $z_g = z_{\text{max}}$.

In Fig. 3d we assume that all the condensate formed below a certain height $z_s$ instantaneously precipitates to the ground, while all the condensate formed at larger heights $z \approx z_s$ is lifted up to $z_{\text{max}}$. In this case, $H_x$ becomes

$$H_x = \frac{\gamma^a(z_1) - \gamma^a(z_s)}{\gamma^a(z_1)} H_P + \frac{\gamma^a(z_s)}{\gamma^a(z_1)} z_{\text{max}}, \quad \text{(A14)}$$

where $H_P$ is given by Eq. (5) with $z_2 = z_s$. The mean cloud water path $\sigma_x$ in this case is

$$\sigma_x = \int_{z_s}^{z_{\text{max}}} \rho_{cc}(z) \, dz. \quad \text{(A15)}$$

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